On Bullen's and related inequalities [1]

Ana Maria Acu, Heiner Gonska

Dedicated to the late Academician Professor Dr. Dimitrie D. Stancu

Abstract

The estimate in Bullen's inequality will be extended for continuous functions using the second order modulus of smoothness. A different form of this inequality will be given in terms of the least concave majorant. Also, the composite case of Bullen's inequality is considered.

1 Motivation

Over the years it happened during several editions of RoGer - the Romanian-German Seminar on Approximation Theory - that the second author learned about inequalities the validity of which was known for regular (i.e., differentiable) functions only. Using tools from Approximation Theory, we showed in [1] and [2] that such restriction can sometimes be dropped and that the estimates can be extended to (at least) continuous functions on the given compact intervals, for example.

Our more general estimates in [1] and [2] were given in terms of the least concave majorant of the usual first order modulus of continuity. Already this is rather a complicated quantity. There we focussed on Ostrowski- and Grüss-type directions. The best way seems to be that via a certain K-functional. This road was recently and thoroughly described in a paper by Păltănea [10]. But the knowledge about this method is much older. See papers by Peetre [11], Mitjagin and Semenov [9] as well as the diploma thesis of Sperling [12], for example.

In Sections 2 and 3 we will consider classical and composite Bullen functionals. It will become clear there that it is quite natural to use the second order modulus of smoothness of a continuous function and a related K-functional.

At the end of this note we will return to the concave majorant and its significance in the field of Inequalities. For the composite case and continuously differentiable functions Section 4 contains a very precise estimate in terms of the majorant.

2   Bullen's inequality revisited

This paper is mostly meant to be a contribution to the ever-lasting discussion on an inequality given by Bullen in [3] (see also an earlier paper by Hammer [7]) in the following form:

Theorem A. *If f is convex and integrable, then*

$$\left(\int_{-1}^{1} f\right) - 2f(0) \le f(-1) + f(1) - \int_{-1}^{1} f.$$

If transformed to an arbitrary compact interval $[a,b] \subset \mathbb{R}$, $a < b$, the equivalent form of the inequality reads

$$\frac{2}{b-a} \int_a^b f(x)dx \le \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right).$$

This is the form we learned about at "RoGer 2014 - Sibiu". In the talk of Petrică Dicu we also learned that in 2000 Dragomir and Pearce [4] had given the following inequality for functions f in $C^2[a,b]$ with known bounds for f'':

Theorem 1 *Let $f : [a,b] \to R$ be a twice differentiable function for which there exist real constants m and M such that*

$$m \le f''(x) \le M, \quad for\ all \quad x \in [a,b].$$

Then

(1)
$$\begin{aligned}
m\frac{(b-a)^2}{24} &\le& \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x)dx \\
&\le& M\frac{(b-a)^2}{24}.
\end{aligned}$$

If we define the Bullen functional B by

$$B(f) := \frac{f(a) + f(b)}{2} + f\left(\frac{a+b}{2}\right) - \frac{2}{b-a} \int_a^b f(x)dx,$$

we note that B is defined for all functions in $C[a,b]$ and that - so far - we have the following

Proposition 1 *The Bullen functional $B : C[a, b] \to \mathbb{R}$ satisfies*

(i) $|B(f)| \leq 4\|f\|_\infty$ *for all $f \in C[a, b]$, $\| \cdot \|_\infty$ indicating the sup norm on $[a, b]$.*

(ii) $|B(g)| \leq \dfrac{(b - a)^2}{24} \cdot \|g''\|_\infty$ *for all $g \in C^2[a, b]$.*

We will next explain how to turn this into a more general statement using the following result from [6]:

Theorem 2 *Let $(F, \| \cdot \|)_F)$ be a Banach space, and let $H : C[a, b] \to (F, \| \cdot \|_F)$ be an operator, where*

a) $\|H(f + g)\|_F \leq \gamma(\|Hf\|_F + \|Hg\|_F)$ *for all $f, g \in C[a, b]$;*

b) $\|Hf\|_F \leq \alpha \cdot \|f\|_\infty$ *for all $f \in C[a, b]$;*

c) $\|Hg\|_F \leq \beta_0 \cdot \|g\|_\infty + \beta_1 \cdot \|g'\|_\infty + \beta_2 \cdot \|g''\|_\infty$ *for all $g \in C^2[a, b]$.*

Then for all $f \in C[a, b]$, $0 < h \leq (b - a)/2$, the following inequality holds:

$$\|Hf\|_F \leq \gamma \left\{ \beta_0 \cdot \|f\|_\infty + \frac{2\beta_1}{h} \omega_1(f; h) + \frac{3}{4} \left(\alpha + \beta_0 + \frac{2\beta_1}{h} + \frac{2\beta_2}{h^2} \right) \omega_2(f; h) \right\}.$$

Taking $H = B$ and $F = \mathbb{R}$ in Theorem 2 we have the following list of constants:

$$\gamma = 1, \quad \alpha = 4, \quad \beta_0 = 0, \quad \beta_1 = 0, \quad \beta_2 = \frac{(b - a)^2}{24}.$$

This takes us to the following

Proposition 2 *For the Bullen functional B and all $f \in C[a, b]$ we have*

$$|B(f)| \leq \left(3 + \left(\frac{b - a}{4h} \right)^2 \right) \cdot \omega_2(f, h), \quad 0 < h \leq \frac{b - a}{2}.$$

The special choice $h = \dfrac{b - a}{k}$, $k \geq 2$ yields

$$|B(f)| \leq \left(3 + \frac{k^2}{16} \right) \cdot \omega_2 \left(f, \frac{b - a}{k} \right).$$

Remark 1 *For $f \in C^2[a, b]$ the latter inequality implies*

$$|B(f)| \leq \left(\frac{1}{16} + \frac{3}{k^2} \right) (b - a)^2 \|f''\|_\infty, k \geq 2.$$

As far as the constant $\left(\dfrac{1}{16} + \dfrac{3}{k^2} \right)$ is concerned, this is much worse than what was invested for C^2 functions.

An alternative estimate is given in the next proposition. Note that it also follows from Theorem 6 in Gavrea's paper [5].

Proposition 3 *If the second K-functional on $C[a,b]$ is defined by*

$$K(f; t^2; C[a,b], C^2[a,b]) := \inf\{||f - g||_\infty + t^2||g''||_\infty : g \in C^2[a,b]\}, \quad t \geq 0,$$

then

$$|B(f)| \leq 4K\left(f; \frac{(b-a)^2}{96}; C[a,b], C^2[a,b]\right).$$

Proof. In [1] the following result was obtained:

$$\left|\frac{1}{2}[(x-a)f(a) + (b-a)f(x) + (b-x)f(b)] - \int_a^b f(t)dt\right|$$

$$\leq 2(b-a)K\left(f; \frac{(x-a)^3 + (b-x)^3}{24(b-a)}; C[a,b], C^2[a,b]\right).$$

The proposition is proved if we substitute $x = \dfrac{a+b}{2}$ in the above inequality.

Remark 2   *(i) For $h \in C^2[a,b]$, we have*

$$
\begin{aligned}
|B(h)| &\leq& 4 \cdot K\left(h; \frac{(b-a)^2}{96}; C[a,b], C^2[a,b]\right) \\
&=& 4 \cdot \inf\left\{||h - g||_\infty + \frac{(b-a)^2}{96}||g''||_\infty, g \in C^2[a,b]\right\} \\
&\leq& 4 \cdot \frac{(b-a)^2}{96}||h''||_\infty \ (taking\ g=h) \\
&=& \frac{(b-a)^2}{24}||h''||_\infty, \quad i.e.,
\end{aligned}
$$

the original inequality for C^2 functions.

(ii) It is known that, for $f \in C[a,b]$,

$$K(f; t^2; C[a,b], C^2[a,b]) \leq c \cdot \omega_2(f, t), \quad 0 \leq t \leq \frac{b-a}{2}$$

with a constant $c \neq c(f, t)$. According to our knowledge the best possible value of c is unknown.

Zhuk showed in [13] that, for $t \leq \dfrac{b-a}{2}$, one has

$$K(f; t^2; C[a,b], C^2[a,b]) \leq \frac{9}{4} \cdot \omega_2(f; t).$$

Using the latter we arrive, for $h \in C^2[a,b]$, at

$$|B(h)| \leq \frac{3(b-a)^2}{32}||h''||_\infty.$$

3   The composite case

Here we consider the composite case of the Bullen functional, i.e., the functional which arises when comparing the composite trapezoidal and midpoint rules. To this end the interval $[a, b]$ is divided in $n \geq 1$ subintervals as follows

$$a = x_0 < \cdots < x_i < x_{i+1} < \cdots < x_n = b.$$

Let the composite Bullen functional $B_c : C[a, b] \to \mathbb{R}$ be given by

$$B_c(f) = \frac{1}{b - a} \sum_{i=0}^{n-1} (x_{i+1} - x_i) \left[\frac{f(x_i) + f(x_{i+1})}{2} + f\left(\frac{x_i + x_{i+1}}{2} \right) \right.$$
$$\left. - \frac{2}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} f(x)dx \right].$$

Proposition 4 *In the composite case there holds*

(i) $|B_c(f)| \leq 4\|f\|_\infty$ *for all* $f \in C[a, b]$,

(ii) $|B_c(g)| \leq \dfrac{1}{24(b - a)} \displaystyle\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \|g''\|_\infty$ *for all* $g \in C^2[a, b]$.

Using Theorem 2 and Proposition 4 we obtain the following inequality for the composite Bullen functional involving the second modulus of continuity:

Proposition 5 *For the composite Bullen functional one has*

$$|B_c(f)| \leq \left(3 + \frac{1}{16(b - a)h^2} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \right) \omega_2(f, h), \ 0 < h \leq \frac{b - a}{2}.$$

For $h = \dfrac{1}{k} \sqrt{\dfrac{\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3}{b - a}}$, $k \geq 2$, *this yields*

$$|B_c(f)| \leq \left(3 + \frac{k^2}{16} \right) \omega_2 \left(f, \frac{1}{k} \sqrt{\frac{\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3}{b - a}} \right), \ k \geq 2.$$

Remark 3 *For* $f \in C^2[a, b]$ *the latter inequality implies*

$$|B_c(f)| \leq \left(\frac{1}{16} + \frac{3}{k^2} \right) \frac{\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3}{b - a} \|f''\|_\infty \leq \left(\frac{1}{16} + \frac{3}{k^2} \right) (b - a)^2 \|f''\|_\infty.$$

The requirement $F(x_0, x_1, \ldots, x_n) = \displaystyle\sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \to minimum$ *entails* $x_{i+1} - x_i = \dfrac{b - a}{n}$, $i = \overline{0, n - 1}$.

The inequality involving a K-functional is given next.

Proposition 6 *For the functional B_c given as above, $f \in C[a,b]$, we have*

$$(2) \qquad |B_c(f)| \leq 4K\left(f; \frac{1}{96(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3; C[a,b], C^2[a,b]\right).$$

Proof. Let $g \in C^2[a,b]$ arbitrary. Then, for $f \in C[a,b]$,

$$|B_c(f)| \leq |B_c(f-g)| + |B_c(g)|$$

$$\leq 4\|f-g\|_\infty + \frac{1}{24(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty$$

$$= 4\left\{\|f-g\|_\infty + \frac{1}{96(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\|g''\|_\infty\right\}.$$

Passing to the infimum over g yields inequality (2).

4 Composite Bullen functional for $f \in C^1[a,b]$

Using the least concave majorant of the modulus of continuity in this section we consider Bullen's inequality for $f \in C^1[a,b]$.

Proposition 7 *If $f \in C^1[a,b]$, then*

$$|B_c(f)| \leq \tilde{\omega}\left(f', \frac{1}{24(b-a)}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3\right).$$

Proof. We have

$$|B_c(f)| \leq \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2} + f\left(\frac{x_i+x_{i+1}}{2}\right)\right.$$

$$\left.- \frac{2}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|$$

$$= \frac{1}{b-a}\sum_{i=0}^{n-1}(x_{i+1}-x_i)\left|\frac{f(x_i)+f(x_{i+1})}{2} - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right.$$

$$\left.+ f\left(\frac{x_i+x_{i+1}}{2}\right) - \frac{1}{x_{i+1}-x_i}\int_{x_i}^{x_{i+1}}f(x)dx\right|$$

$$= \frac{1}{b-a}\sum_{i=0}^{n-1}\left|\int_{x_i}^{x_{i+1}}\left(\frac{f(x_i)-f(x)}{2} + \frac{f(x_{i+1})-f(x)}{2}\right.\right.$$

$$\left.\left.+ f\left(\frac{x_i+x_{i+1}}{2}\right) - f(x)\right)dx\right| \leq 2\|f'\|_\infty.$$

Let $g \in C^2[a,b]$. Using Proposition 4 and the latter inequality implies

$$|B_c(f)| = |B_c(f - g + g)| \leq |B_c(f - g)| + |B_c(g)|$$

$$\leq 2\|(f - g)'\|_\infty + \frac{1}{24(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \|g''\|_\infty$$

$$= 2 \left\{ \|(f - g)'\|_\infty + \frac{1}{48(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3 \|g''\|_\infty \right\}.$$

Passing to the infimum over $g \in C^2[a,b]$ we have

$$|B_c(f)| \leq 2K \left(f'; \frac{1}{48(b-a)} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^3; C^1[a,b], C^2[a,b] \right),$$

so the result follows as a consequence of the relation (see [10])

$$K \left(f'; t; C^1[a,b], C^2[a,b] \right) = \frac{1}{2} \tilde{\omega}(f', 2t), 0 \leq t \leq \frac{b-a}{2}.$$

A particular consequence of Proposition 7 is the following version of Bullen's inequality.

Proposition 8 *If $f \in C^1[a,b]$, then*

$$|B(f)| \leq \tilde{\omega} \left(f', \frac{(b-a)^2}{24} \right).$$

References

[1] A. Acu, H. Gonska, *Ostrowski inequalities and moduli of smoothness*, Results Math. 53 (2009), 217-228.

[2] A. Acu, H. Gonska, I. Raşa, *Grüss- and Ostrowski-type inequalities in Approximation Theory*, Ukrain. Mat. Zh. 63 (2011), 723-740, and Ukrainian Math. J. 63 (2011), 843-864.

[3] P.S. Bullen, *Error estimates for some elementary quadrature rules*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. No. 602-633 (1978), 97-103 (1979).

[4] S.S. Dragomir, Ch.E.M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGNMIA Monographs, Victoria University, 2002 (amended version).

[5] I. Gavrea, *Preservation of Lipschitz constants by linear transformations and global smoothness preservation*, Functions, Series, Operators (L.Leindler, F. Schipp, J. Szabados, eds.), Budapest, 2002, pp. 261-275.

[6] H. Gonska, R. Kovacheva, *The second order modulus revisited: remarks, applications, problems*, Confer. Sem. Mat. Univ. Bari No. 257 (1994), 32 pp. (1995).

[7] P.C. Hammer, *The midpoint method of numerical integration*, Math. Mag. 31 (1957/1958), 193–195.

[8] N. Minculete, P. Dicu, A. Raţiu, *Two reverse inequalities of Bullen's inequality and several applications*, Lecture given at RoGer 2014 - Sibiu.

[9] B.S. Mitjagin, M.E. Semenov, *Absence of interpolation of linear operators in spaces of smooth functions* (Russian), Izv. Akad. Nauk SSSR Ser. Mat. 41 (1977), no. 6, 1289–1328, 1447.

[10] R. Pǎltǎnea, Representation of the K-functional $K(f, C[a,b], C^1[a,b], \cdot)$ - a new approach. Bull. Transilv. Univ. Braşov Ser. III 3(52) (2010), 93–99.

[11] J. Peetre, On the connection between the theory of interpolation spaces and approximation theory. Proc. Conf. Constructive Theory of Functions (Approximation Theory) (Budapest, 1969), 351–363. Akademiai Kiado, Budapest, 1972.

[12] A. Sperling, *Konstanten in den Sätzen von Jackson und in den Ungleichungen zwischen K-Funktionalen und Stetigkeitsmoduln*, Diplomarbeit, Universität Duisburg, 1984.

[13] V.V. Zhuk, *Functions of the Lip1 class and S. N. Bernstein's polynomials*, (Russian). Vestnik Leningrad. Univ. Mat. Mekh. Astronom. 1989, vyp. 1, 25–30, 122–123, Vestnik Leningrad Univ. Math. 22 (1989), no. 1, 38-44.

Ana Maria Acu
Lucian Blaga University of Sibiu
Faculty of Sciences
Department of Mathematics and Informatics
Str. Dr. I. Ratiu, No.5-7, 550012 Sibiu, Romania
e-mail: acuana77@yahoo.com

Heiner Gonska
University of Duisburg-Essen
Faculty of Mathematics
Forsthausweg 2, 47057 Duisburg, Germany
e-mail: heiner.gonska@uni-due.de